Subject: Report on the doctoral dissertation of Mr. Goran Radunović, Department of Applied Mathematics, FER, University of Zagreb

The Faculty Council appointed us at the session held on January 14th 2015 for giving an opinion about the doctoral dissertation of Mr. Goran Radunović under the title

**FRACTAL ANALYSIS OF UNBOUNDED SETS IN EUCLIDEAN SPACES AND LAPIDUS ZETA FUNCTIONS**

Hereby we submit our

**REPORT**

The proposed dissertation has approximately 250 pp., and it consists of six chapters, accompanied with the bibliography (more than 150 references), the curriculum vitae (in English and Croatian), as well as with 15 figures.

**Motivation**

The higher-dimensional theory of fractal zeta functions was initiated by M. L. Lapidus in 2009 and developed in cooperation with G. Radunović and D. Žubrinić in

An overview of the monograph is available in the following reference:


[LapRaŽu1] provides a significant extension of the theory of geometric zeta functions of bounded fractal strings (which are essentially one-dimensional objects) to the case of bounded subsets of Euclidean spaces of arbitrary dimension. Bounded fractal strings and the associated geometric (and spectral) zeta functions, as well as the theory of complex dimensions of fractal strings, have been introduced and studied by M. L. Lapidus and his collaborators over the past more than two decades (more precisely, since the early 1990's) in numerous research papers and in several monographs. We mention in particular the following extensive monograph:


In [LapRaŽu1], the central role is played by the distance zeta functions of arbitrary (nonempty) bounded subsets of Euclidean spaces of any dimension, discovered by M. L. Lapidus, as well as of more general objects called the relative fractal drums (RFDs), which contain bounded fractal strings as a special case. The proposed zeta functions exhibit deep connections with the upper box dimension of the set (or of the RFD) under consideration, as well as with the upper and lower Minkowski contents. More precisely, the abscissa of (absolute) convergence of the Lapidus zeta function is always equal to the upper box dimension of the set under consideration, while (under some mild conditions), its residue computed at the abscissa of convergence is (up to a constant factor) always between the upper and lower Minkowski contents of the set. A similar result holds for the so-called tube zeta functions, defined via the tube function of a fractal set.

Historically, a particular attention to the notion of Minkowski content arose in connection with the (modified) Weyl-Berry conjecture from Physics, which has been completely resolved in the one-dimensional case by M. L. Lapidus and C. Pommerance in 1993. Another impetus came from a very surprising result due to M. L. Lapidus and H. Maier in 1995, that the famous Riemann hypothesis can be formulated in terms of the so-called inverse spectral problem for fractal strings, involving the notion of Minkowski measurability.

For these reasons, it is clear that the fractal zeta functions (by which we mean distance, tube, spectral and geometric zeta functions), as well as their relation to the box dimensions and Minkowski contents of bounded sets (and of RFDs), deserve a special attention. Another completely unexpected novelty is that for RFDs, the associated relative Minkowski dimension can assume negative values as well, including minus infinity. This is related to the flatness property of the relative fractal drum, described and studied in Chapter 4 of [LapRaŽu1].
The motivation to study the fractal properties of unbounded sets comes from a variety of sources. In particular, the notion of "unbounded" or "divergent" oscillations appears in problems in oscillation theory, automotive industry, civil engineering and mathematical applications in biology. Furthermore, unbounded domains themselves are also interesting in the theory of elliptic partial differential equations. More precisely, the question of solvability of the Dirichlet problem for quasilinear equations in general unbounded domains is addressed by A. Mazja.

Goals

The main goal of the thesis is to provide a potentially useful extension of the theory of fractal zeta functions of bounded subsets of Euclidean spaces, to the case of unbounded subsets.

First of all, the problem of finding (pointwise and distributional) fractal tube formulas for RFDs is addressed, aiming to view in a new light and in much greater generality the corresponding results in the context of fractal strings, obtained in [Lap-vFr3] and in the relevant references cited therein. This set of problems is studied in Chapter 3.

Another goal of the present thesis is to extend the theory of fractal zeta functions of bounded subsets of Euclidean spaces of any dimension to the case unbounded subsets, viewed as relative fractal drums with respect to infinity. To this end, it was necessary to define in a suitable way the corresponding fractal zeta functions, as well as the Minkowski contents and box dimensions of unbounded sets. These problems are treated in Chapter 4.

It is natural to study the fractal properties of trajectories of polynomial vector fields at infinity. This is done for weak foci of some classes of planar vector fields (and bifurcation problems), like those appearing during the classical Hopf (or, more generally, the Hopf-Takens) bifurcation.

A short overview of the content of the thesis

The introductory Chapter 1 provides an overview of the main results of the thesis, as well as of the basic definitions and notation. A special emphasis is placed on the Dirichlet-type integrals (DTIs) and on the almost periodic functions, needed in later chapters.

Chapter 2 also has an introductory character, aiming to provide a rapid overview of the main definitions and results of the theory of Lapidus zeta functions, both for bounded sets and of relative fractal drums in Euclidean spaces, which was initiated in 2009.

Chapter 3 deals with fractal tube formulas for relative fractal drums in Euclidean spaces. A particular attention is paid on pointwise and distributional tube formulas for RFDs, as well as on the problem of characterization of their Minkowski measurability. Some of the results from Chapter 3 appear in the following survey article:

Chapter 4 introduces the notion of Lapidus zeta functions at infinity. The problem of finding their abscissa of convergence is resolved, as well as the problem of the construction of their meromorphic continuation, both in a Minkowski measurable and a Minkowski nonmeasurable case. Also, the problem of constructing quasiperiodic unbounded sets is studied, and infinite classes of such sets have been constructed which are $n$-quasiperiodic, for any integer $n$ larger than 1, including even $n$ equal to infinity.

Chapter 5 studies the fractal properties of unbounded trajectories of polynomial planar vector fields which have weak focus at infinity, as well as of the Hopf-Takens bifurcation at infinity. Its content is based on following, already published joint work:


A very short Chapter 6 provides concluding remarks and discusses some of the perspectives of the theory. In the Appendix, the author proposed a set of thirteen open problems for future research.

Overview of the main results

Chapter 3 deals with an important problem of obtaining fractal tube formulas associated with RFDs in Euclidean spaces. Roughly speaking, the problem is to derive the asymptotics of the relative tube function from knowing the complex dimensions of the RFD under consideration, i.e., the poles of a suitable meromorphic extension of the tube zeta function of a given RFD. These formulas hold either pointwise or in the sense of the Schwartz theory of distributions, depending on the growth properties of the corresponding relative tube zeta function. They generalize the corresponding results obtained in [Lap-vFr3] for bounded fractal strings.

Fractal tube formulas have been obtained by representing the relative tube zeta function in the form of the Mellin transform of a (suitably modified) relative tube function of the RFD under consideration. A special care is taken about pointwise and distributional error estimates in fractal tube formulas in the case of relative tube zeta functions. These results are then “translated” in terms of relative distance zeta functions. The translation was not immediate. In order to do so, the author had to introduce a new type of fractal zeta functions, called the relative shell zeta functions. The most interesting results are stated in Theorems 3.37 and 3.40.

As we have already mentioned, of particular value in the dissertation is a characterization of Minkowski measurability of a large class of RFDs, expressed (under some mild conditions) in terms of its relative distance function. More precisely, the Minkowski measurability of an RFD belonging to this class is equivalent to the requirement that the value $D$ of the relative box dimension of the RFD is well defined, that $D$ is the only complex dimension of the RFD located on the critical line of absolute convergence, and finally, that it is simple. The proof of this result is technically rather demanding, and a part of it is based on a Tauberian theorem.
due to Wiener and Pitt which generalizes the famous Ikehara's Tauberian theorem. The characterization of Minkowski measurability is first proved in the case of $D < N$ (Theorem 3.14), and then for $D = N$ (Theorem 3.86). The proof in the latter case uses a very interesting idea of embedding the problem into a Euclidean space of higher dimension.

The author has applied his new fractal tube formulas to $a$-strings, to the ternary Cantor string and to the Fibonacci string, retrieving already known results from [Lap-vFr3], as well as to the Sierpiński gasket studied by M. L. Lapidus and E. P. J. Pearse in 2010. Of course, Radunović has obtained as a byproduct some new interesting examples of fractal tube formulas, like for the 3-Sierpiński carpet (known also as the Sierpiński cube), for a class of unbounded geometric chirps depending on two parameters, for an RFD based on the Cantor function graph, for a class of fractal nests etc. Moreover, he was able to study a class of RFDs with degenerate relative Minkowski contents, using suitable gauge functions (Theorems 3.73 and 3.74).

An important and very general explicit formula has been obtained for the relative distance zeta function of a class of self-similar sprays generated by finitely many similarity ratios and satisfying the usual open set condition. It is stated in Equation (3.6.79), and the formula extends the corresponding ones obtained in [Lap-vFr]. This formula also enabled the author to recover well-known fractal tube formulas for self-similar sprays with monophase or pluriphase generators obtained by M. L. Lapidus, E. P. J. Pearse and S. Winter in 2011 and 2012.

Moreover, this formula enabled the author to construct a class of RFDs, such that the corresponding principal complex dimensions (viewed as the poles of the associated relative distance zeta function, suitably meromorphically extended) can be of arbitrarily high order (Example 3.76). The idea was to use iterated fractal sprays of finite order, starting by a suitably chosen base fractal spray. Nontrivial examples are obtained already for the ternary Cantor spray as the base spray. Taking iterated fractal sprays of infinite order, one can construct an RFD possessing even essential singularities on the critical line of its relative distance zeta function.

Some of the results of this chapter are announced in the following preprint:


Chapter 4 represents the second main contribution of this thesis to the higher-dimensional theory of complex dimensions and fractal zeta functions. The author introduced the notions of Minkowski content and box dimension of unbounded sets at infinity for unbounded sets of finite Lebesgue measure, and derived their basic properties. Then he introduced the notion of the Lapidus (or distance) zeta function at infinity, which is of fundamental importance. It is then shown in Theorem 4.24 that the abscissa of its absolute convergence is equal to the upper box dimension of the unbounded set at infinity under consideration, contained in a given $N$-dimensional Euclidean space.

The value of the dimension is always negative, and moreover, it is less than or equal to $-N$. In Theorem 4.21, the author showed that the Lapidus zeta function at infinity is in fact equal to
the usual distance zeta function of the geometrically inverted set with respect to the origin. However, as the author stressed, it is not immediately evident what are the relations between the usual relative box dimension and Minkowski content (in the sense of [LapRaŽu1]) of the inverted relative fractal drum on the one side and the notions of box dimension and Minkowski content at infinity introduced in Chapter 4 on the other.

After introducing the tube zeta function at infinity, the functional equation relating it to the Lapidus zeta function at infinity is derived in Theorem 4.31. The problem of the existence and construction of a meromorphic continuation for these new classes of fractal zeta functions is solved in the Minkowski measurable and the Minkowski non-measurable case, respectively (under some additional conditions), in Theorems 4.36 and 4.39. A sufficient condition for Minkowski measurability at infinity is established in Theorem 4.38 by means of the Wiener-Pitt Tauberian theorem, i.e., using methods developed in Chapter 3.

The property of quasiperiodicity of unbounded sets at infinity has been studied by means of number-theoretic methods. In Theorem 4.75 it has been proved that algebraically $n$-quasiperiodic unbounded sets can be constructed for any integer $n$ greater than 1. Here, $n$ has the meaning of the number of quasiperiods. The same theorem contains an analogous result for transcendentally $n$-quasiperiodic unbounded sets. It is also possible to treat the case of infinitely many quasiperiods, and the corresponding result is obtained in Theorem 4.81. The proofs rest, among others, on deep results from analytic number theory due to S. I. Besicovitch and A. Baker. It is worth mentioning here that the existence of an algebraically quasiperiodic bounded set is an open problem. The author has also studied the problem of constructing maximally hyperfractal sets at infinity, i.e., such that the associated fractal zeta functions at infinity have the whole critical line as their natural boundary.

In Section 4.7 the author explored the connection between the fractal properties of unbounded sets at infinity and their images on the Riemann sphere; that is, he explored the effect of the one-point compactification on the fractal properties of unbounded sets. He introduced the notions of spherical Minkowski content and spherical box dimension and provided a general result in Theorem 4.84 that connects them to the notions of Minkowski content at infinity and box dimension at infinity, respectively. Inspired by the fact that the spherical Minkowski content is well defined even for unbounded sets of infinite Lebesgue measure, in Definition 4.86 he introduced a new notion of a parametric Minkowski content which he calls the shell Minkowski content and which depends on a real parameter. The corresponding results dealing with Lapidus zeta functions at infinity for sets of possibly infinite Lebesgue measure are obtained Theorems 4.117, 4.121 and 4.122.

Chapter 5 is basically the incorporation of the above mentioned research paper [RaŽuŽup] into the broader context of the thesis. It deals with vector fields possessing spiral trajectories tending to infinity. The provided visualizations clearly show that in the case of a weak focus at infinity such trajectories exhibit an almost “planar” nature. This phenomenon can be measured using the box dimension of trajectories. Since the trajectories tending to infinity are unbounded, it is necessary to adapt the definition of box dimension to this case, since the usual box dimension is defined for bounded sets only. This can be done by using the geometric inversion. It is shown that every polynomial vector field $P$ can be geometrically inverted; that is, there exists a polynomial vector field $Q$ such that its phase portrait is equal to the geometric inversion of the phase portrait $P$ (Lemma 5.23).
A new result not appearing in the paper is given in Theorem 5.5, which connects the upper and lower Minkowski contents of a set near the origin with its upper and lower spherical Minkowski contents. A connection between the two approaches to fractal analysis of unbounded sets (that is, between the approach via the geometric inversion of Chapter 5 and the approach via ‘fractality at infinity’ of Chapter 4) is established in Theorem 5.19. The effect of the Poincaré compactification on the fractal properties of a focus type spirals is examined in Section 5.4.

Several examples of vector fields are provided having a weak focus at infinity, including the example of the classical Hopf bifurcation (Theorem 5.34), while in Section 5.7 the more general Hopf-Takens bifurcation at infinity is considered (Theorems 5.35 and 5.36).
CONCLUSION

The thesis of Goran Radunović is an extensive and high quality work, written with great care and erudition. It belongs to a very attractive area bordering complex analysis and fractal geometry. The author obtained, among others, pointwise and distributional fractal tube formulas for a class of relative fractal drums (RFDs) in Euclidean spaces and studied fractal properties of unbounded sets, viewed as RFDs with respect to infinity, including their Minkowski measurability and quasi-periodicity. Alongside, he introduced and studied the new notions of a parametric Minkowski content and of fractal shell zeta functions for RFDs. The dissertation opens new perspectives for further research in several directions, ranging from the theory of complex dimensions of fractal strings to recently introduced Lapidus zeta functions of relative fractal drums in Euclidean spaces of any dimension, as well as to dynamical systems.

The candidate has mastered and successfully applied very diverse and nontrivial mathematical techniques, ranging from complex analysis and Tauberian theory to fractal geometry, distribution theory, analytic number theory, as well as to bifurcation theory of planar vector fields. He illustrated his results on carefully chosen examples. The overall content of the dissertation, along with its rigorous and clear style of exposition, indicate the scientific maturity of the author.

For these reasons, we recommend the acceptance of the thesis

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(FRAKTALNA ANALIZA NEOMEĐENIH SKUPOVA U EUKLIDSKIM PROSTORIMA I LAPIDUSOVE ZETA FUNKCIJE)

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